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The $\mathcal{U}$-Radius for Classes of Analytic Functions

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Abstract Let $\mathcal{U}$ denote the class of normalized analytic functions f in the open unit disk $\mathbb{D}$ satisfying

$$\left|\left(\frac{z}{f(z)}\right)^2 f'(z) - 1\right| < 1.$$

The $\mathcal{U}$-radius is obtained for several classes of functions. These include the class of normalized analytic functions f satisfying the inequality $\operatorname{Re}\, f(z)/g(z) > 0$ or $|f(z)/g(z) - 1| < 1$ in $\mathbb{D}$, where g belongs to a certain class of functions, the class of functions f satisfying $|f'(z) - 1| < 1$ in $\mathbb{D}$, and functions f satisfying $\operatorname{Re}\, f(z)/z > \alpha$, $0 \le \alpha < 1$, in $\mathbb{D}$. A recent conjecture by Obradović and Ponnusamy concerning the radius of univalence for a product involving univalent functions is also shown to hold true.

Keywords Analytic functions · Univalent functions · Convex functions · Starlike functions · $\mathcal{U}$-radius

Mathematics Subject Classification Primary: 30C45 · 30C50

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1 Introduction

Let $\mathcal{A}$ denote the class of analytic functions f in $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ normalized by $f(0) = 0 = f'(0) - 1$. Let $\mathcal{S}$ be its subclass consisting of univalent functions. Denote by $\mathcal{S}^*$ and $\mathcal{C}$ the subclasses of $\mathcal{S}$ consisting, respectively, of starlike (with respect to the origin) and convex functions. Geometrically $f \in \mathcal{S}^*$ if the linear segment $tw, 0 \le t \le 1$, lies completely in $f(\mathbb{D})$ whenever $w \in f(\mathbb{D})$, while $f \in \mathcal{C}$ if $f(\mathbb{D})$ is a convex domain. These functions are, respectively, characterized analytically by $\mathrm{Re}\ \left(zf'(z)/f(z)\right) > 0$ and $\mathrm{Re}\ \left(1 + zf''(z)/f'(z)\right) > 0$.

For $0 \le \alpha < 1$, let $\mathcal{P}(\alpha)$ denote the class of analytic functions p satisfying $p(0) = 1$ and $\mathrm{Re}\ p(z) > \alpha$ in $\mathbb{D}$, with $\mathcal{P} := \mathcal{P}(0)$. Thus $f \in \mathcal{S}^*$ is equivalent to $zf'(z)/f(z) \in \mathcal{P}$. Likewise, $f \in \mathcal{C}$ if $1 + zf''(z)/f'(z) \in \mathcal{P}$.

Let $\mathcal{U}$ denote the subclass consisting of functions $f \in \mathcal{A}$ satisfying $|\mathcal{U}_f(z)| < 1$ for $z \in \mathbb{D}$, where

$$\mathcal{U}_f(z) = \left(\frac{z}{f(z)}\right)^2 f'(z) - 1.$$

As early as 1958, Aksent'ev [1] showed that functions in $\mathcal{U}$ are univalent in $\mathbb{D}$. However the converse need not hold, as illustrated by the convex function $f(z) = -\log(1-z)$. Evidently $\left|\mathcal{U}_f(z)\right| > 1$ for real z close to 1. Though functions in $\mathcal{U}$ need not be starlike [5,17], the Koebe function $k(z) = z/(1-z)^2$ is an important example of a function in $\mathcal{U} \cap \mathcal{S}^*$. Indeed each function in the set

$$S_{\mathbb{Z}} = \left\{z,\ \frac{z}{(1 \pm z)^2},\ \frac{z}{1 \pm z},\ \frac{z}{1 \pm z^2},\ \frac{z}{1 \pm z + z^2}\right\}$$

belongs to $\mathcal{U}$. Interestingly, functions in $S_{\mathbb{Z}}$ are known [6] to be the only functions in $\mathcal{S}$ with integer coefficients in their series expansions. Thus $S_{\mathbb{Z}} \subset \mathcal{U} \cap \mathcal{S}^* \subset \mathcal{S}$.

Functions $f \in \mathcal{U}$ have a close connection with the class Σ consisting of univalent meromorphic functions F in $\Delta := \left\{\zeta : |\zeta| > 1\right\} \cup \{\infty\}$ with $F(\zeta) \neq 0$ and of the form

$$F(\zeta) = \zeta + \sum_{n=0}^{\infty} c_n \zeta^{-n}, \quad \zeta \in \Delta.$$

Indeed the correspondence is given by

$$F(\zeta) = \frac{1}{f(1/\zeta)}, \quad \zeta \in \Delta,$$

and the change of variable $\zeta = 1/z$ readily yields

$$F'(\zeta) - 1 = f'(1/\zeta)/\left(\zeta^2 f^2(1/\zeta)\right) - 1 = \mathcal{U}_f(z).$$

The class $\mathcal{U}$ has been widely studied in recent years, for example in the works of [14–22] and [25]. Several interesting properties of the class $\mathcal{U}$ are shaped by the coefficients of its mappings. If $f \in \mathcal{S}$, then $z/f(z)$ is nonvanishing in $\mathbb{D}$ and has a series representation of the form

$$\frac{z}{f(z)} = 1 + \sum_{n=1}^{\infty} b_n z^n. \tag{1.1}$$

It follows from the area theorem [7, Theorem 11, p. 193] that

$$\sum_{n=2}^{\infty} (n-1)|b_n|^2 \leq 1. \tag{1.2}$$

Obradović and Ponnusamy [19] showed that every $f \in \mathcal{A}$ of the form (1.1) belongs to the class $\mathcal{U}$ whenever $\sum_{n=2}^{\infty}(n-1)|b_n| \leq 1$. They [20] also showed that $f(z) = z + \sum_{n=2}^{\infty} a_n z^n \in \mathcal{A}$ satisfying $\sum_{n=2}^{\infty} n|a_n| \leq 1$ belongs to $\mathcal{U} \cap \mathcal{S}^*$. On the other hand, it was shown in [2] that functions $f \in \mathcal{U}$ of the form (1.1) necessarily satisfy $\sum_{n=2}^{\infty}(n-1)^2|b_n|^2 \leq 1$.

In [2], Ali et al. showed that the condition (1.2) does not ensure univalence, and they obtained the sharp radius of univalence $r_0 = 1/\sqrt{2}$ for the class of functions $f \in \mathcal{A}$ satisfying (1.2). In [16], the $\mathcal{U}$-radius for $\mathcal{S}$ was determined to be $1/\sqrt{2}$. Evidently, radius problems have continued to be an important area of study.

In general, for two families $\mathcal{G}$ and $\mathcal{F}$ of $\mathcal{A}$, the $\mathcal{G}$-*radius* for the class $\mathcal{F}$, denoted by $R_{\mathcal{G}}(\mathcal{F})$, is the largest number R such that $r^{-1} f(rz) \in \mathcal{G}$ for $0 < r \leq R$, and $f \in \mathcal{F}$.

In [12, 13], MacGregor obtained the radius of starlikeness for the class of functions $f \in \mathcal{A}$ satisfying either

$$\operatorname{Re}\left(\frac{f(z)}{g(z)}\right) > 0 \quad (z \in \mathbb{D}) \quad \text{or} \quad \left|\frac{f(z)}{g(z)} - 1\right| < 1 \quad (z \in \mathbb{D}) \tag{1.3}$$

for some $g \in \mathcal{C}$. Ratti [27] determined its radius of starlikeness for the class (1.3) when g belongs to certain classes of analytic functions. MacGregor in [11] also found the radius of convexity for univalent functions satisfying $|f'(z) - 1| < 1$.

This paper finds the $\mathcal{U}$-radius for three classes of functions:

(a) first is the class of functions $f \in \mathcal{A}$ satisfying the inequality

$$\operatorname{Re}\left(\frac{f(z)}{g(z)}\right) > 0, \quad z \in \mathbb{D}, \tag{1.4}$$

for some $g \in \mathcal{A}$ with

$$\operatorname{Re}\left(\frac{g(z)}{z}\right) > 0, \quad z \in \mathbb{D};$$

(b) secondly the class of functions $f \in \mathcal{A}$ satisfying the inequality

$$\operatorname{Re}\left(\frac{f(z)}{g(z)}\right) > 0, \quad z \in \mathbb{D}, \tag{1.5}$$

for some $g \in \mathcal{A}$ with

$$\operatorname{Re}\left(\frac{g(z)}{z}\right) > \frac{1}{2}, \quad z \in \mathbb{D};$$

(c) and the class of functions $f \in \mathcal{A}$ satisfying the inequality

$$\left|\frac{f(z)}{g(z)} - 1\right| < 1, \quad z \in \mathbb{D},$$

for some $g \in A$ with

$$\operatorname{Re}\left(\frac{g(z)}{z}\right) > 0, \quad z \in \mathbb{D}.$$

Additionally, this paper also finds the radius r_0 so that

$$\left|\mathcal{U}_f(z)\right| = \left|\left(\frac{z}{f(z)}\right)^2 f'(z) - 1\right| < 1$$

in the disk $|z| < r_0$ for the following two classes of functions:

(a) the subclass of close-to-convex functions $f \in \mathcal{A}$ satisfying

$$\left|f'(z) - 1\right| < 1, \quad z \in \mathbb{D}; \tag{1.6}$$

(b) and the class of functions $f \in \mathcal{A}$ satisfying the inequality

$$\operatorname{Re}\frac{f(z)}{z} > \alpha, \quad 0 \le \alpha < 1, \quad z \in \mathbb{D}. \tag{1.7}$$

It is known that every convex function in $\mathcal{C}$ belongs to the class (1.7) for $\alpha = 1/2$. Indeed, this class also contains $f \in \mathcal{U}$ satisfying $f''(0) = 0$.

Ratti [27] showed that the radius of starlikeness for the class (1.4) is $\sqrt{5} - 2$, and that the radius can be improved to 1/3 for the class given by (1.5). The radius of convexity for the class given by (1.6) was obtained by MacGregor [11]. Several radius constants, which include the radius of starlikeness of a positive order, radius of parabolic starlikeness, radius of Bernoulli lemniscate starlikeness, and radius of uniform convexity, have been obtained for the classes (1.4) and (1.5) by Ali et al. in [3].

Obradović and Ponnusamy in [21] also considered the product of functions $F(z) = f(z)g(z)/z$ when f and g belong to certain subsets of $\mathcal{S}$. They showed that whenever $f, g \in \mathcal{S}^*$, then the product F is starlike in the disk $|z| < 1/3$. Additionally, F belongs to $\mathcal{U}$ in the disk $|z| < r_0$, where $r_0 \approx 0.30294$, whenever $f, g \in \mathcal{S}$. In [22], they improved the value of r_0 to $r_0 \approx 0.3263$, where r_0 is the positive root of a certain equation. When $f, g \in \mathcal{S}$, they [21] conjectured that F is also univalent in the disk $|z| < 1/3$, and that the radius $1/3$ is best. In Sect. 3, we show in the affirmative this conjecture. Indeed, the radius of starlikeness for such functions F is shown to be $1/3$.

The following lemmas are needed in the sequel. Recall that an analytic function f is subordinate to an analytic function g, written $f(z) \prec g(z)$, if there exists an analytic self-map w of $\mathbb{D}$ with $w(0) = 0$ satisfying $f(z) = g\big(w(z)\big)$.

Lemma 1.1 [9] *Let $p(z) = 1 + p_1 z + \cdots$ be analytic in $\mathbb{D}$, and h be convex. If*

$$p(z) + \frac{1}{\gamma} z p'(z) \prec h(z), \tag{1.8}$$

where $\gamma \neq 0$ and $\operatorname{Re} \gamma \geq 0$, then

$$p(z) \prec \frac{\gamma}{z^\gamma} \int_0^z h(t) t^{\gamma - 1} dt.$$

Lemma 1.2 [18] *Let f be analytic in $\mathbb{D}$ and have the form*

$$\frac{z}{f(z)} = 1 + b_1 z + b_2 z^2 + \cdots ,$$

with $b_n \geq 0$ for all $n \geq 2$. Then the following are equivalent:

(a) $f \in \mathcal{S}$,

(b) $\frac{f(z) f'(z)}{z} \neq 0, \quad z \in \mathbb{D}$,

(c) $\sum_{n=2}^{\infty} (n-1) b_n \leq 1$,

(d) $f \in \mathcal{U}$.

2 The $\mathcal{U}$-Radius for Classes of Analytic Functions

Theorem 2.1 *The $\mathcal{U}$-radius for the class of functions $f \in \mathcal{A}$ satisfying the inequality*

$$\operatorname{Re} \left(\frac{f(z)}{g(z)} \right) > 0, \quad z \in \mathbb{D},$$

for some $g \in \mathcal{A}$ with

$$\operatorname{Re} \left(\frac{g(z)}{z} \right) > 0, \quad z \in \mathbb{D},$$

is $r_{\mathcal{U}} = \sqrt{5} - 2 \approx 0.23607$.

Proof Writing $p(z) = g(z)/z$ and $h(z) = f(z)/g(z)$, it follows that $p, h \in \mathcal{P}$ and $f(z) = z p(z) h(z)$. A brief computation shows that

$$\begin{aligned}\mathcal{U}_f(z) &= z^2\frac{f'(z)}{f^2(z)} - 1\\ &= \frac{1}{h(z)}\left(\frac{1}{p(z)} - z\left(\frac{1}{p(z)}\right)' - 1\right) + \frac{1}{p(z)}\left(\frac{1}{h(z)} - z\left(\frac{1}{h(z)}\right)' - 1\right)\\ &\quad - \left(\frac{1}{p(z)} - 1\right)\left(\frac{1}{h(z)} - 1\right).\end{aligned}$$

Thus

$$\begin{aligned}|\mathcal{U}_f(z)| &\le \left|\frac{1}{h(z)}\right|\left|\frac{1}{p(z)} - z\left(\frac{1}{p(z)}\right)' - 1\right| + \left|\frac{1}{p(z)}\right|\left|\frac{1}{h(z)} - z\left(\frac{1}{h(z)}\right)' - 1\right|\\ &\quad + \left|\frac{1}{h(z)} - 1\right|\left|\frac{1}{p(z)} - 1\right|. \qquad (2.1)\end{aligned}$$

Since $1/p(z) = 1 + \sum_{n=1}^{\infty} b_n z^n$ and $1/h(z) = 1 + \sum_{n=1}^{\infty} c_n z^n$ are in the class $\mathcal{P}$, then $|b_n| \le 2$ and $|c_n| \le 2$ for $n \ge 1$. Thus for $|z| = r$,

$$\left|\frac{1}{p(z)} - 1\right| \le \sum_{n=1}^{\infty} |b_n||z|^n \le 2\sum_{n=1}^{\infty} r^n = \frac{2r}{1-r}, \qquad (2.2)$$

and

$$\left|\frac{1}{p(z)} - z\left(\frac{1}{p(z)}\right)' - 1\right| \le \sum_{n=2}^{\infty}(n-1)|b_n||z|^n \le 2\sum_{n=2}^{\infty}(n-1)r^n = \frac{2r^2}{(1-r)^2}. \qquad (2.3)$$

Similar estimates are obtained for the function $1/h$.

From (2.1), it follows that

$$|\mathcal{U}_f(z)| \le 2\left(\frac{1+r}{1-r}\frac{2r^2}{(1-r)^2}\right) + \frac{4r^2}{(1-r)^2} = \frac{8r^2}{(1-r)^3}.$$

Hence $|\mathcal{U}_f(z)| < 1$ if $|z| < \sqrt{5} - 2$, where $r_{\mathcal{U}} = \sqrt{5} - 2$ is the root of the equation $(r+1)(r^2 + 4r - 1) = 0$.

To demonstrate sharpness, let $f_0(z) = z\big((1-z)/(1+z)\big)^2$, and $g_0(z) = z(1-z)/(1+z)$. Evidently

$$\begin{aligned}|\mathcal{U}_{f_0}(r)| &= \left|\left(\frac{r}{f_0(r)}\right)^2 f_0{}'(r) - 1\right|\\ &= \left|\left(\frac{(1+r)^2}{(1-r)^2}\right)^2\left(\frac{(1-r)^2}{(1+r)^2} - \frac{4r(1-r)}{(1+r)^3}\right) - 1\right|\end{aligned}$$

$$= \left| \frac{(1+r)^2(1-r) - 4r(1+r)}{(1-r)^3} - 1 \right|$$
$$= \frac{8r^2}{(1-r)^3}.$$

Since $r^2/(1-r)^3$ is increasing, it follows that $|\mathcal{U}_{f_0}(r)| > 1$ for $\sqrt{5} - 2 < r < 1$. □

Theorem 2.2 *The $\mathcal{U}$-radius for the class of functions $f \in \mathcal{A}$ satisfying the inequality*

$$\operatorname{Re}\left(\frac{f(z)}{g(z)}\right) > 0, \quad z \in \mathbb{D},$$

for some $g \in \mathcal{A}$ with

$$\operatorname{Re}\left(\frac{g(z)}{z}\right) > \frac{1}{2}, \quad z \in \mathbb{D},$$

is $r_{\mathcal{U}} = 1/3$.

Proof Let $p(z) = g(z)/z, h(z) = f(z)/g(z)$, and $f(z) = zp(z)h(z)$. Since $p \in \mathcal{P}(1/2)$, it follows that $p(z) \prec 1/(1+z)$, and thus $p(z) = 1/\big(1 + z\varphi(z)\big)$, where $|\varphi(z)| \le 1$. For $|z| = r$, and $|\varphi(z)| = x$, $0 \le x \le 1$, evidently

$$\left| \frac{1}{p(z)} - 1 \right| = |z||\varphi(z)| = rx,$$

and the Schwarz–Pick inequality [4, p. 198] gives

$$\left| \frac{1}{p(z)} - z\left(\frac{1}{p(z)}\right)' - 1 \right| = |z|^2|\varphi'(z)| \le \frac{|z|^2\big(1 - |\varphi(z)|^2\big)}{1 - |z|^2} = \frac{r^2(1-x^2)}{1-r^2}.$$

The function $h \in \mathcal{P}$ satisfies the estimates (2.2) and (2.3). It follows from (2.1) that

$$|\mathcal{U}_f(z)| \le \frac{1+r}{1-r}\frac{r^2(1-x^2)}{1-r^2} + \frac{2r^2(1+rx)}{(1-r)^2} + \frac{2r^2x}{1-r} = \frac{r^2(3+2x-x^2)}{(1-r)^2}.$$

Since $\lambda(x) = 3 + 2x - x^2$ is increasing over $0 \le x \le 1$, evidently

$$|\mathcal{U}_f(z)| \le \frac{4r^2}{(1-r)^2} < 1$$

if $r < r_{\mathcal{U}}$, where $r_{\mathcal{U}} = 1/3$ is the root of the equation $3r^2 + 2r - 1 = 0$.

For sharpness, consider $f_0(z) = z(1-z)/(1+z)^2$, and $g_0(z) = z/(1+z)$. Thus

$$\frac{z}{f_0(z)} = \frac{(1+z)^2}{1-z} = 1 + 3z + 4\sum_{n=2}^{\infty} z^n.$$

It follows from Lemma 1.2 that $r^{-1} f_0(rz) \in \mathcal{U}$ provided $0 < r \leq 1$ satisfies

$$4\sum_{n=2}^{\infty}(n-1)r^n = 4r\sum_{n=1}^{\infty} nr^n = \left(\frac{2r}{1-r}\right)^2 \leq 1,$$

that is, if $r \leq r_{\mathcal{U}}$, where $r_{\mathcal{U}} = 1/3$ is the root of the equation $3r^2 + 2r - 1 = 0$. □

Theorem 2.3 *The $\mathcal{U}$-radius for the class of functions $f \in \mathcal{A}$ satisfying the inequality*

$$\left|\frac{f(z)}{g(z)} - 1\right| < 1, \quad z \in \mathbb{D},$$

for some $g \in A$ with

$$\operatorname{Re}\left(\frac{g(z)}{z}\right) > 0, \quad z \in \mathbb{D},$$

is $r_{\mathcal{U}} = \left(\sqrt{17} - 3\right)/4 \approx 0.28078$.

Proof Let $p(z) = g(z)/z$, $h(z) = f(z)/g(z)$, and $f(z) = zp(z)h(z)$. Then $1/h(z) = 1 + \sum_{n=1}^{\infty} c_n z^n$ is in $\mathcal{P}(1/2)$, and thus $|c_n| \leq 1$ for $n \geq 1$. For $|z| = r$,

$$\left|\frac{1}{h(z)} - 1\right| \leq \sum_{n=1}^{\infty} |c_n||z|^n \leq \sum_{n=1}^{\infty} r^n = \frac{r}{1-r},$$

and

$$\left|\frac{1}{h(z)} - z\left(\frac{1}{h(z)}\right)' - 1\right| \leq \sum_{n=1}^{\infty}(n-1)|c_n||z|^n \leq \sum_{n=1}^{\infty}(n-1)r^n = \frac{r^2}{(1-r)^2}.$$

Further the function p satisfies the estimates (2.2) and (2.3). It follows from (2.1) that

$$|\mathcal{U}_f(z)| \leq \frac{1}{1-r}\frac{2r^2}{(1-r)^2} + \frac{1+r}{1-r}\frac{r^2}{(1-r)^2} + \frac{r}{1-r}\frac{2r}{1-r} = \frac{5r^2 - r^3}{(1-r)^3}.$$

Hence $|\mathcal{U}_f(z)| < 1$ if $r < r_{\mathcal{U}}$, where $r_{\mathcal{U}} = \left(\sqrt{17} - 3\right)/4$ is the root of the equation $2r^2 + 3r - 1 = 0$.

To demonstrate sharpness, let $f_0(z) = z(1-z)^2/(1+z)$, and $g_0(z) = z(1-z)/(1+z)$. Thus

$$\frac{z}{f_0(z)} = \frac{1+z}{(1-z)^2} = 1 + \sum_{n=1}^{\infty}(2n+1)z^n.$$

Lemma 1.2 will be used to show $r^{-1}f_0(rz) \in \mathcal{U}$. For $0 < r \leq 1$,

$$\frac{rz}{f_0(rz)} = 1 + \sum_{n=1}^{\infty}(2n+1)r^n z^n,$$

and

$$\begin{aligned}\sum_{n=2}^{\infty}(n-1)(2n+1)r^n &= 2r^2\sum_{n=0}^{\infty}(n+2)(n+1)r^n + r^2\sum_{n=0}^{\infty}(n+1)r^n \\ &= \frac{4r^2}{(1-r)^3} + \frac{r^2}{(1-r)^2} = \frac{5r^2-r^3}{(1-r)^3} \leq 1\end{aligned}$$

if and only if $r \leq \left(\sqrt{17}-3\right)/4$, where $r_{\mathcal{U}} = \left(\sqrt{17}-3\right)/4$ is the root of the equation $2r^2+3r-1=0$. □

Theorem 2.4 *Let $f \in \mathcal{A}$ satisfy*

$$|f'(z)-1| < 1, z \in \mathbb{D}.$$

Then

$$\left|\mathcal{U}_f(z)\right| = \left|\left(\frac{z}{f(z)}\right)^2 f'(z) - 1\right| < 1$$

in the disk $|z| < r_0$, where $r_0 = \sqrt{(\sqrt{5}-1)/2} \approx 0.78615$.

Proof Evidently the subordination (1.8) translates to $f'(z) \prec 1+z$ by choosing $\gamma = 1$, $p(z) = f(z)/z$, and $h(z) = 1+z$. It follows that

$$\frac{f(z)}{z} \prec 1 + \frac{z}{2}.$$

Thus there exists an analytic self-map w of $\mathbb{D}$ with $w(0) = 0$ and $f(z)/z = 1+w(z)/2$.

Simple computations lead to

$$\begin{aligned}|\mathcal{U}_f(z)| &= \left|\left(\frac{1}{1+\frac{w(z)}{2}}\right)^2\left(1+\frac{w(z)}{2}+\frac{zw'(z)}{2}\right)-1\right| \\ &= \frac{\left|2\left(zw'(z)-w(z)\right)-w^2(z)\right|}{4\left|1+\frac{w(z)}{2}\right|^2}.\end{aligned}$$

Thus

$$|\mathcal{U}_f(z)| \leq \frac{2\left|zw'(z)-w(z)\right|+|w^2(z)|}{4\left(1-\frac{|w(z)|}{2}\right)^2}. \tag{2.4}$$

The Schwarz–Pick inequality [4, p. 198] applied to $w(z)/z$ yields

$$|zw'(z) - w(z)| \leq \frac{|z|^2 - |w(z)|^2}{1 - |z|^2}. \tag{2.5}$$

Substituting (2.5) into (2.4), and writing $|w(z)| = t$, $|z| = r$, $0 \leq t \leq r$, leads to

$$\begin{aligned} |\mathcal{U}_f(z)| &\leq \frac{1}{\left(1 - r^2\right)} \left(\frac{-\left(1 + r^2\right) t^2 + 2r^2}{(2 - t)^2} \right) \\ &:= \frac{1}{1 - r^2} \Phi(t, r). \end{aligned}$$

Since

$$\frac{\partial \Phi(t, r)}{\partial t} = \frac{4\left(r^2 - (1 + r^2)t\right)}{(2 - t)^3},$$

the function $\Phi(t, r)$ attains its maximum at the point $t_0 = r^2/(1 + r^2)$, that is,

$$\Phi(t, r) \leq \frac{r^2\left(r^2 + 1\right)}{\left(r^2 + 2\right)}.$$

Thus $|\mathcal{U}_f(z)| < 1$ if $|z| < r_0$, where $r_0 = \sqrt{(\sqrt{5} - 1)/2}$ is the root of the equation $r^4 + r^2 - 1 = 0$. □

Remark 2.1 Ozaki [23] introduced the class $\mathcal{G}$ consisting of functions $f \in \mathcal{A}$ satisfying

$$\operatorname{Re}\left(1 + \frac{zf''(z)}{f'(z)}\right) < \frac{3}{2},$$

and proved that these functions are necessarily univalent in $\mathbb{D}$. Umezawa [30] showed that these functions are convex in one direction. Sakaguchi [28] proved that $|\arg f'(z)| < \pi/2$ whenever $f \in \mathcal{G}$, and indeed, $\mathcal{G} \subset \mathcal{S}^*$, see [10,29]. There has been a continued interest in recent years over the class $\mathcal{G}$, see for example, the works in [24,26]. It follows from [10, Theorem 2] that $|f'(z) - 1| < 1$ whenever $f \in \mathcal{G}$. Thus Theorem 2.4 shows that $\left|\mathcal{U}_f(z)\right| < 1$ for $f \in \mathcal{G}$, and $|z| < \sqrt{(\sqrt{5} - 1)/2}$.

Theorem 2.5 *Let $f \in \mathcal{A}$ satisfy*

$$\operatorname{Re} \frac{f(z)}{z} > \alpha, \quad 0 \leq \alpha < 1, \quad z \in \mathbb{D}.$$

Then

$$\left|\mathcal{U}_f(z)\right| = \left|\left(\frac{z}{f(z)}\right)^2 f'(z) - 1\right| < 1$$

in the disk $|z| < r(\alpha)$, *where*

$$r(\alpha) = \begin{cases} \sqrt{\frac{2(1-\alpha)}{1-2\alpha}} - 1, & 0 \le \alpha \le \frac{1}{10}, \\ \sqrt{\frac{\sqrt{\alpha(1-\alpha)}-\alpha}{1-2\alpha}}, & \frac{1}{10} \le \alpha \le \frac{1}{2}, \\ \sqrt{\frac{\sqrt{2-4\alpha(1-\alpha)}-2(1-\alpha)}{2(2\alpha-1)}}, & \frac{1}{2} \le \alpha \le \tau_\alpha, \\ \frac{1}{4\alpha-3}\left(1 - \sqrt{\frac{2(1-\alpha)}{2\alpha-1}}\right), & \tau_\alpha \le \alpha < 1, \end{cases}$$

and $\tau_\alpha = \left(8 - 11/\sqrt[3]{71+6\sqrt{177}} + \sqrt[3]{71+6\sqrt{177}}\right)/12 \approx 0.93804$ *is the root of the equation*

$$4\alpha - \left((2\alpha-1) + \sqrt{(2\alpha-1)(10\alpha-9)}\right)\left((2a-1) + \sqrt{2(1-\alpha)(2\alpha-1)}\right) = 2 \tag{2.6}$$

in the interval $[9/10, 1)$. *The result is sharp for the case* $\alpha \in [0, 1/10]$.

Proof It follows that

$$\frac{f(z)}{z} \prec \alpha + (1-\alpha)\frac{1+z}{1-z}.$$

Thus there exists an analytic self-map w of $\mathbb{D}$ satisfying $w(0) = 0$ and

$$f(z) = z\left(\frac{1 + (1-2\alpha)w(z)}{1 - w(z)}\right).$$

Now

$$\mathcal{U}_f(z) = \frac{2(1-\alpha)\left(\left(zw'(z) - w(z)\right) - (1-2\alpha)w^2(z)\right)}{(1 + (1-2\alpha)w(z))^2},$$

and

$$|\mathcal{U}_f(z)| \le \frac{2(1-\alpha)\left(\left|zw'(z) - w(z)\right| + |1-2\alpha||w(z)|^2\right)}{(1 - |1-2\alpha||w(z)|)^2}.$$

Writing $|w(z)| = t$, $|z| = r$, $0 \le t \le r$, and $|1-2\alpha| = a$, it follows from (2.5) that

$$\begin{aligned} |\mathcal{U}_f(z)| &\le \frac{2(1-\alpha)\left(r^2 + \left(a(1-r^2) - 1\right)t^2\right)}{(1-at)^2(1-r^2)} \\ &:= \Phi(t, r). \end{aligned}$$

Evidently

$$\frac{\partial \Phi(t,r)}{\partial t} = \frac{4(1-\alpha)\left(ar^2 - \left(1 - a(1-r^2)\right)t\right)}{(1-r^2)(1-at)^3} := \frac{4(1-\alpha)}{(1-r^2)(1-at)^3}\varphi(t,r),$$

where

$$\varphi(t,r) = ar^2 - \Big(1 - a(1-r^2)\Big)\,t.$$

Thus the critical points of $\Phi(t,r)$ over $t \in [0,r]$ occurs at $t = 0,\ t = r$ and possibly at $t_0 = ar^2/(1 - a + ar^2)$, where $\varphi(t_0, r) = 0$. Indeed t_0 is a critical point in $[0, r]$ whenever

$$g(r) = ar^2 - ar + 1 - a \tag{2.7}$$

is nonnegative.

For $a \in [0, 4/5]$, it is evident that $g(r) \geq 0$ in $(0, 1)$. Hence the maximum value of $\Phi(t,r)$ is the largest value of $\{\Phi(0,r); \Phi(t_0,r); \Phi(r,r)\}$. Since

$$\begin{aligned}\Phi(t_0,r) - \Phi(0,r) &= \frac{2(1-\alpha)(1-a+ar^2)r^2}{(1-a)(1-r^2)(1+ar^2)} - \frac{2(1-\alpha)r^2}{(1-r^2)} \\ &= \frac{2a^2(1-\alpha)r^4}{(1-a)(1-r^2)(1+ar^2)} \geq 0,\end{aligned}$$

and

$$\begin{aligned}\Phi(t_0,r) - \Phi(r,r) &= \frac{2(1-\alpha)(1-a+ar^2)r^2}{(1-a)(1-r^2)(1+ar^2)} - \frac{2(1-\alpha)ar^2}{(1-ar)^2} \\ &= \frac{2r^2(1-\alpha)\big((1-a) - ar(1-r)\big)^2}{(1-a)(1-r^2)(1+ar^2)(1-ar)^2} \geq 0,\end{aligned}$$

it is evident that max $\Phi(t,r) = \Phi(t_0,r)$. Thus

$$|\mathcal{U}_f(z)| \leq \Phi(t_0,r) = \frac{2(1-\alpha)(1-a+ar^2)r^2}{(1-a)(1-r^2)(1+ar^2)} < 1$$

provided $|z| < r(\alpha)$, where $r(\alpha)$ is the root of the equation

$$a\big(3 - 2\alpha - a\big)r^4 + (1-a)\big(3 - 2\alpha - a\big)r^2 - (1-a) = 0.$$

Hence

$$r^2(\alpha) = \frac{\sqrt{(1-a)^2\big(3-2\alpha-a\big)^2 + 4a(1-a)\big(3-2\alpha-a\big)} - (1-a)\big(3-2\alpha-a\big)}{2a\big(3-2\alpha-a\big)}.$$

Since $a = |1 - 2\alpha|$, further simplification leads to

$$r(\alpha) = \begin{cases} \sqrt{\frac{\sqrt{\alpha(1-\alpha)}-\alpha}{1-2\alpha}}, & \frac{1}{10} \leq \alpha \leq \frac{1}{2} \\ \sqrt{\frac{\sqrt{2-4\alpha(1-\alpha)}-2(1-\alpha)}{2(2\alpha-1)}}, & \frac{1}{2} \leq \alpha \leq \frac{9}{10}. \end{cases}$$

For $a \in [4/5, 1]$, the roots of g in (2.7) occurs at

$$r_1 = \frac{a - \sqrt{a(5a-4)}}{2a}, \quad r_2 = \frac{a + \sqrt{a(5a-4)}}{2a}. \tag{2.8}$$

Evidently $g(r) \geq 0$ over the intervals $[0, r_1]$ and $[r_2, 1)$, and so the maximum of $\Phi(t, r)$ occurs at $\Phi(t_0, r)$. On the other hand, $g(r) < 0$ over (r_1, r_2). Since t_0 is not a critical point, the maximum of $\Phi(t, r)$ occurs at either $\Phi(0, r)$ or $\Phi(r, r)$.

Consider

$$\begin{aligned} K(r) = \Phi(0, r) - \Phi(r, r) &= \frac{2(1-\alpha)r^2}{(1-r^2)} - \frac{2(1-\alpha)ar^2}{(1-ar)^2} \\ &= \frac{2r^2(1-\alpha)\big(a(1+a)r^2 - 2ar + 1 - a\big)}{(1-r^2)(1-ar)^2} \\ &:= \frac{2r^2(1-\alpha)}{(1-r^2)(1-ar)^2}k(r), \end{aligned}$$

where

$$k(r) = a(1+a)r^2 - 2ar + 1 - a, \quad a \in [4/5, 1].$$

The roots of k are

$$r_1' = \frac{a - \sqrt{a^3 + a(a-1)}}{a(1+a)}, \quad r_2' = \frac{a + \sqrt{a^3 + a(a-1)}}{a(1+a)}.$$

Observe that $K(r) \leq 0$ over (r_1', r_2'), and that $(r_1, r_2) \subseteq (r_1', r_2')$, where r_1, r_2 are given by (2.8). Thus $K(r) \leq 0$ over (r_1, r_2), and the maximum value of $\Phi(t, r)$ is $\Phi(r, r)$.

There are two cases to consider for $a = |1 - 2\alpha| \in [4/5, 1]$, that is, $\alpha \in [0, 1/10]$ and $\alpha \in [9/10, 1)$. Consider first when $\alpha \in [0, 1/10]$.

If $r \in [0, r_1]$, then g given by (2.7) satisfies $g(r) \geq 0$. Thus

$$|\mathcal{U}_f(z)| \leq \Phi(t_0, r) < 1$$

for all $|z| < R_1(\alpha)$, where

$$R_1(\alpha) = \sqrt{\frac{\sqrt{\alpha(1-\alpha)} - \alpha}{1 - 2\alpha}}$$

is the root of the equation $\Phi(t_0, r) = 1$. Since $R_1(\alpha) \geq r_1$, it follows that

$$|\mathcal{U}_f(z)| < 1$$

whenever $|z| < r_1$.

When $r \in (r_1, r_2)$, then $g(r) < 0$ and

$$|\mathcal{U}_f(z)| \leq \Phi(r, r) < 1$$

for all $|z| < R_2(\alpha)$, where

$$R_2(\alpha) = \sqrt{\frac{2(1-\alpha)}{1-2\alpha}} - 1$$

is the root of the equation $\Phi(r, r) = 1$. Since $r_1 < R_2(\alpha) < r_2$, we deduce that $|\mathcal{U}_f(z)| < 1$ for all $|z| < R_2(\alpha)$ when $\alpha \in [0, 1/10]$.

Consider next the other case when $\alpha \in [9/10, 1)$. Likewise as in the first case, if $r \in [0, r_1]$, then $g(r) \geq 0$ and

$$|\mathcal{U}_f(z)| \leq \Phi(t_0, r) < 1$$

for all $|z| < R_1'(\alpha)$, where

$$R_1'(\alpha) = \sqrt{\frac{\sqrt{2 - 4\alpha(1-\alpha)} - 2(1-\alpha)}{2(2\alpha - 1)}} \tag{2.9}$$

is the root of the equation $\Phi(t_0, r) = 1$. Since $R_1'(\alpha) \geq r_1$, then

$$|\mathcal{U}_f(z)| < 1$$

whenever $|z| < r_1$.

If $r \in (r_1, r_2)$, then $g(r) < 0$ and

$$|\mathcal{U}_f(z)| \leq \Phi(r, r) < 1$$

for all $|z| < R_2'(\alpha)$, where

$$R_2'(\alpha) = \frac{1}{4\alpha - 3}\left(1 - \sqrt{\frac{2(1-\alpha)}{2\alpha - 1}}\right)$$

is the root of the equation $\Phi(r, r) = 1$.

A closer scrutiny of $R_2'(\alpha)$ reveals that $r_1 < R_2'(\alpha) < r_2$ whenever $\alpha \in [\tau_\alpha, 1)$, where τ_α is given by (2.6). Thus in this case, $|\mathcal{U}_f(z)| < 1$ for $|z| < R_2'(\alpha)$.

On the other hand, $R_2'(\alpha) \geq r_2$ whenever $\alpha \in [9/10, \tau_\alpha]$. Thus if $r \in [r_2, 1)$, then $g(r) \geq 0$ and

$$|\mathcal{U}_f(z)| \leq \Phi(t_0, r) < 1$$

for all $|z| < R_1'(\alpha)$, where $R_1'(\alpha)$ is given by (2.9). Since $r_2 \leq R_1'(\alpha) < 1$, it follows that $|\mathcal{U}_f(z)| < 1$ for $|z| < R_1'(\alpha)$ when $\alpha \in [9/10, \tau_\alpha]$.

For $\alpha \in [0, 1/10]$, an extremal function is $f_0(z) = z\big(1-(1-2\alpha)z\big)/(1+z)$. In this case,

$$\begin{aligned} \frac{z}{f_0(z)} &= \frac{(1+z)}{1-(1-2\alpha)z} \\ &= 1 + 2(1-\alpha)\sum_{n=1}^{\infty}(1-2\alpha)^{n-1}z^n. \end{aligned}$$

Thus

$$\begin{aligned} \frac{Rz}{f_0(Rz)} &= 1 + 2(1-\alpha)\sum_{n=1}^{\infty}(1-2\alpha)^{n-1}R^n z^n \\ &= 1 + \sum_{n=1}^{\infty} b_n z^n. \end{aligned}$$

for $0 < R \leq 1$. Evidently

$$\begin{aligned} \sum_{n=2}^{\infty}(n-1)b_n &= 2(1-\alpha)(1-2\alpha)R^2\sum_{n=2}^{\infty}(n-1)\big((1-2\alpha)R\big)^{n-2} \\ &= \frac{2(1-\alpha)(1-2\alpha)R^2}{\big(1-(1-2\alpha)R\big)^2} \leq 1 \end{aligned}$$

if and only if $R \leq R(\alpha)$, where $R(\alpha) = \sqrt{2(1-\alpha)/(1-2\alpha)} - 1$ is the root of the equation

$$(1-2\alpha)R^2 + 2(1-2\alpha)R - 1 = 0.$$

It follows from Lemma 1.2 that $R^{-1}f_0(Rz) \in \mathcal{U}$ if $R \leq \sqrt{2(1-\alpha)/(1-2\alpha)} - 1$. $\square$

3 Product of Univalent Functions

Let $\mathcal{F}_1$ and $\mathcal{F}_2$ be subsets of $\mathcal{S}$. In [21], Obradović and Ponnusamy considered functions

$$F(z) = \frac{f(z)g(z)}{z}, \quad z \in \mathbb{D}, \tag{3.1}$$

where $f \in \mathcal{F}_1$ and $g \in \mathcal{F}_2$. If f and g are in $\mathcal{U}$, they showed that F defined by (3.1) belongs to $\mathcal{U}$ in the disk $|z| < 1/3$, and that this radius is sharp. Indeed if $f, g \in \mathcal{S}^*$, then the product F is starlike in the disk $|z| < 1/3$. When $f, g \in \mathcal{S}$, they conjectured that F is univalent in the disk $|z| < 1/3$, and that this radius is best. Here we shall validate the conjecture.

Lemma 3.1 *If $f \in \mathcal{S}$, then*

$$\mathrm{Re}\left(\frac{zf'(z)}{f(z)}\right) > \frac{1-r}{1+r},$$

for $|z| = r < \tanh(1/2) \approx 0.46212$.

Proof Let $f \in \mathcal{S}$. It is known [8] that for $|z| \leq r < 1$, the region of values of $\zeta = \log(zf'(z)/f(z))$ is the disk

$$\mathcal{D}_r = \left\{\zeta : |\zeta| \leq \log\frac{1+r}{1-r}\right\}.$$

The function $w(z) = e^z$ is univalent in $\mathbb{D}$. Thus if r is chosen so that $\log\big((1+r)/(1-r)\big) < 1$, that is, $r < \tanh(1/2)$, then w is univalent in $\mathcal{D}_r$. Evidently the function $q(z) = w(z) - 1$ is convex in $\mathbb{D}$, that is, w is a convex function with positive coefficients in its series expansion. Thus

$$\begin{aligned}\inf_{\zeta\in\mathcal{D}_r} \mathrm{Re}\ w(\zeta) &= \inf_{0\leq\theta\leq 2\pi} \mathrm{Re}\ \exp\left(\log\left(\frac{1+r}{1-r}\right)\cos\theta\right)\\ &= \exp\left(-\left(\log\frac{1+r}{1-r}\right)\right) = \frac{1-r}{1+r}\end{aligned}$$

for $|z| = r < \tanh(1/2)$. □

Theorem 3.1 *If $f, g \in \mathcal{S}$, then the function F defined by* (3.1) *is starlike in the disk* $|z| < 1/3$. *The radius* $1/3$ *is sharp.*

Proof It follows from (3.1) that

$$\mathrm{Re}\left(\frac{zF'(z)}{F(z)}\right) = \mathrm{Re}\left(\frac{zf'(z)}{f(z)}\right) + \mathrm{Re}\left(\frac{zg'(z)}{g(z)}\right) - 1.$$

Lemma 3.1 now shows that F is starlike when $|z| < 1/3$. Sharpness is demonstrated by letting $f(z) = z/(1-z)^2 = g(z)$. □

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